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Derivation of Green's Functions of Complex Structures Using Computer Algebra

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The manual implementation of existing formulations for the algebraic derivation of Green's functions of complex interconnected structures based on the Green's functions of the component structures can be very tedious. The objective of this investigation is to explore the application of computer algebra to this problem. For simple structures such as uniform Euler-Bernoulli beams and flat plates, the Green's functions can be derived directly from the partial differential equations. The more general forms of these functions are expressed as infinite series of terms contributed by the characteristic functions and parameters of the system. In this paper, computer algebra has been used to automate the derivation of the characteristic functions and parameters of the combined structure. Examples of complex structures such as uniform cantilivered Euler-Bernoulli beams to which discrete spring-mass substructures have been attached were used to assess the accuracy of the proposed method in comparison with results of finite element analyses and previous techniques for solving the same problem. The results show that computer algebra derives the correct functions and parameters to any specified accuracy. Moreover, problems of numerical ill conditioning associated with matrix methods are avoided.

Nomenclature

A	= cross-sectional area
EI	= flexural rigidity
$F_c(p)$	= Laplace transform of $f_c(t)$
$f_c(t)$	= applied excitation to substructure
$G(x, \xi, t, \tau)$, $G(x, \xi, t)$	= Green's function
h_k	= objective function
K	= stiffness
K_b	= $3EI/L^3$
L	= length
$L_{x,t}\{\ \}$	= partial differential operator
M	= mass
M_1, M_2	= spatial domains
p	= Laplace variable
$Q(x, t)$	= system response
$Q_c(p)$	= Laplace transform of $q_c(t)$
$q_c(t)$	= response of substructure
t	= time
$W(x, \xi, p)$	= transfer function
$W(\mathrm{d}/\mathrm{d}t)$	= differential operator for substructure
$w(x, t)$	= excitation signal
x, x_1, x_2	= spatial coordinates
α_{kn}	= coefficient of spectral expansion
β^2	= mass per unit length
$\delta(\ )$	= Dirac delta function
$\lambda, \lambda(p)$	= characteristic parameters
ρ	= density
σ	= real part of the Laplace variable
$\psi(x), \psi_k(x)$	= characteristic functions
Ω	= time interval
ω	= imaginary part of the Laplace variable

Superscripts

$\|\|(\ )\|\|$	= Euclidean norm of ( )
$(\ )^*$	= complex conjugate of ( )

Subscripts

$_D(\ )$	= denominator of ( )
$_N(\ )$	= numerator of ( )

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Introduction

LARGE interconnected space structures that are deployed for applications in low-to-zero gravity environments have posed new challenges to structural dynamicists, not so much because of their physical size and the operating environment, but mainly because of the need for precise control of their motions. The stringent specifications on the vibrations (jitter) that can be tolerated during operation make it important to have good analytical models that can facilitate the design of active controllers that are used to suppress unwanted vibrations. Because of the need to include higher order elastic modes in the dynamical analysis, numerical techniques (e.g., finite element method) usually involve large-order matrices that are susceptible to ill conditioning, which may lead to unanticipated losses in accuracy. Alternative approaches that try to avoid the problem of large-order matrices involve the derivation of Green's functions (also known as impulse response functions) for the structural system. These approaches permit the treatment of the structure as a distributed parameter system, thereby avoiding the manipulation of large-order matrices. The response of the structure at a given coordinate due to excitation at another is calculated directly without the inversion of matrices. The forced response of the structure due to an arbitrary excitation force is simply obtained as the convolution integral between the Green's function and the excitation force.

For simple structures such as uniform Euler-Bernoulli beams and flat plates, the Green's functions can be derived directly from the partial differential equations subject to appropriate boundary conditions.[1-3] In some cases, the expression for the Green's function is obtained in a closed but "split" form[1] (i.e., the expression takes different forms for response locations that lie on different sides of the excitation location); however, the more general derivations result in an infinite series of terms contributed by the characteristic functions and parameters of the system. Practical structures are usually

more complicated and, in general, may not lend themselves to direct derivation of the Green's functions. In many instances, it is possible to idealize the real structural system by an interconnection of simpler structures. Many authors have examined the problem of calculating the free vibration modes and frequencies of simple beams with discrete attachment of lumped parameter substructures: Bishop,[1] Rayleigh,[4] Young,[5] Bisplinghoff,[6] Dowell,[7] Nicholson and Bergman,[8] and Broome,[9] to name just a few. In calculating the free vibration modes and frequencies of the combined system, some authors have utilized the Green's function of the unmodified beam.[8-10] These efforts did not address the derivation of the Green's functions of the combined system, which are suitable for the forced response analysis.

General methods for obtaining the Green's functions of interconnected structural systems, using integral equations and generalized functions to combine the Green's functions of the constituent structures, have been published by Butkovskiy.[11] These techniques accept the Green's functions of the substructures in whatever analytical form that they are available. However, because of the algebraic manipulations necessary to obtain the final result, the manual implementations of these methods have been restricted to simple interconnections. In order to deal with more realistic situations, which may involve multiple interconnections, this research has explored the utilization of computer algebraic approaches to this problem. With computer algebra,[12,13] the implementation of Butkovskiy's technique becomes feasible for any baseline structure for which a Green's function is available for any number of arbitrary interconnections, provided the appropriate functions are available, and for any number of arbitrary interconnections, provided the appropriate functions are available for representing the response/excitation relationships. Because the final form of the Green's function of the combined system may be very complicated, it is desirable to have a form for the Green's function that displays the contributions of the characteristic modes of the system and from which these functions and parameters can be extracted when needed. In this paper, the well-studied example of the cantilever beam with a spring-mass attachment has been used to illustrate the proposed approach.

For dynamical systems governed by differential equations of the Sturm-Liouville class, spectral techniques can be used to derive the Green's functions as an infinite series of contributions from the set of basis functions, which are obtained from the homogeneous equation.[14] If the Green's function of the combined dynamical system is assumed to retain this algebraic form, then the expressions obtained by Butkovskiy's method can be manipulated algebraically to yield the basis functions and characteristic parameters that correspond to the combined system. When the combined dynamical system is undamped (i.e., both the baseline structure and the attachments contain no dissipative elements), these basis functions and characteristic parameters yield the free vibration modes and frequencies of the combined structure. But when the baseline structure and/or the attachments include damping, methods that calculate the free vibration modes and frequencies directly (e.g., Nicholson and Bergman[8]) are not applicable, since there are no free vibrations, and the method of separation of variables (which is necessary for such methods to work), is no longer feasible. This problem does not arise if the Green's function of the combined system is obtained first, using the methods proposed by Butkovskiy. The basis functions and characteristic parameters that are obtained using the derivations presented in this paper would then correspond to the complex modes and parameters that would be obtained if a modal parameter extraction were performed for the damped system.

The remainder of this paper has been organized as follows. First the spectral form of the Green's functions for systems governed by equations of the Sturm-Liouville class is presented followed by the result of Butkovskiy's method for obtaining the Laplace transform of the Green's function of a combined dynamical system, which consists of a distributed parameter baseline structure to which a lumped parameter substructure has been attached. The derivations of the algebraic algorithms for obtaining the modified characteristic parameters and characteristic functions are then presented. This is followed by a number of examples that consider the well-studied case of a cantilever beam and a spring-mass attachment at the tip. These examples are used to confirm that the characteristic parameters and functions recovered by these algorithms do indeed agree with those given by Young[5] and the finite element method.

Spectral Form for Green's Functions of Sturm-Liouville Systems

Let the response of a distributed parameter dynamical system to excitation $w(x_1, t)$ be $Q(x_2, t)$; $x_1 \in M_1$, $x_2 \in M_2$, $t \in \Omega$; where M_1 and M_2 are sets of spatial variables of the excitation and response signals, respectively, and Ω is an interval of time (t_0, t_1). The partial differential equation of motion can be written as

$$L_{x,t}\{Q(x_2, t)\} = w(x_1, t) \tag{1}$$

where $L_{x,t}\{\ \}$ is a partial differential operator of the form:

$$L_{x,t}\{\ \} = L\left(x, \frac{\partial}{\partial x}\right)\{\ \} - \lambda\left(\frac{\partial}{\partial t}\right)\{\ \} \tag{2}$$

By defining a Green's function that satisfies the following equation:

$$L_{x,t}\{G(x, \xi, t, \tau)\} = \delta(x - \xi)\delta(t - \tau) \tag{3}$$

the solution to Eq. (1) is given directly by

$$Q(x_2, t) = \int_\Omega \int_{M_1} G(x_2, \xi, t, \tau)\, w(\xi, t)\, d\xi\, d\tau \tag{4}$$

For stationary systems, $G(x, \xi, t, \tau)$ is of the form $G(x, \xi, t - \tau)$, and the Laplace transform of Eq. (3) gives

$$\left[L\left(x, \frac{\partial}{\partial x}\right) - \lambda(p)\right]\{W(x, \xi, p)\} = \delta(x - \xi) \tag{5}$$

where $p = \sigma + i\omega$ is the Laplace variable, and $W(x, \xi, p)$ is the Laplace transform of $G(x, \xi, t - \tau)$. Although Eq.(5) gives $W(x, \xi, p)$ the right to be called a Green's function, it is convenient to distinguish it from $G(x, \xi, t - \tau)$ by using the term "transfer function" since this is consistent with the common practice in control theory. In this paper, as was done in Ref. 11, $G(x, \xi, t - \tau)$ will be called the Green's function and $W(x, \xi, p)$ the transfer function. $G(x, \xi, t - \tau)$ and $W(x, \xi, p)$ form a Laplace transform pair.

For simple structures such as uniform beams, $W(x, \xi, p)$ is available in a closed but "split" form[2]; however, the inverse Laplace transform of this form is not a simple expression. It is of interest to formulate a more general form for $W(x, \xi, p)$ that is not only applicable to a broader class of structures, but also for which the inverse Laplace transform is a straightforward expression. If $\psi_k(x)$ and $\lambda(p_k)$ are the characteristic function and characteristic parameter of the following eigenvalue equation for $k = 1,2, \ldots \infty$ subject to the appropriate boundary conditions

$$L\left(x, \frac{\partial}{\partial x}\right)\{\psi(x)\} = \lambda(p)\psi(x) \tag{6}$$

It has been shown[14,15] that for systems governed by differential equations of the Sturm-Liouville class the transfer function is of the form

$$W(x, \xi, p) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(\xi)}{\lambda(p_k) - \lambda(p)} \tag{7}$$

If, for example, $\lambda(p) = -\beta^2 p^2$, as is the case for a uniform beam where $\beta^2 = \rho A$, the mass per unit length,[16] the Green's function given by the inverse Laplace transform of Eq. (7) is simply

$$G(x, \xi, t) = \frac{1}{\beta^2} \sum_{k=1}^{\infty} \frac{1}{ip_k} \psi_k(x)\psi_k(\xi) \sin ip_k t \tag{8}$$

Transfer Function for Combined Distributed Parameter System with Lumped Parameter Attachment

Consider a lumped parameter substructure governed by the constant coefficient ordinary differential equation:

$$W_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\{q_c(t)\} = f_c(t) \tag{9}$$

where $f_c(t)$ is the applied excitation at the attachment point, $q_c(t)$ the response of the substructure at the attachment point, and $W_c(\mathrm{d}/\mathrm{d}t)$ the differential operator with constant coefficients. By taking the Laplace transform of Eq. (9), an algebraic equation is obtained that can be used to obtain the ratio of the Laplace transforms of the excitation force to the response as

$$\frac{F_c(p)}{Q_c(p)} = W_c(p) \tag{10}$$

where $F_c(p)$ and $Q_c(p)$ are the Laplace transforms of $f_c(t)$ and $q_c(t)$, respectively. If the coordinate of attachment of this substructure to the distributed parameter baseline structure is $x = b$, the new transfer function for the combined system has been shown by Butkovskiy[11] to be

$$W_1(x, \xi, p) = W_0(x, \xi, p) + \frac{W_0(x, b, p)W_0(b, \xi, p)}{1/W_c(p) - W_0(b, b, p)} \tag{11}$$

where $W_0(x, \xi, p)$ is the transfer function of the unmodified structure, and $W_1(x, \xi, p)$ is that of the combined system. The expression given by Eq. (11) is an exact relationship between the baseline transfer function and the transfer function of the combined system. If the baseline transfer function is available in exact form, the transfer function of the combined structure will also be obtained exactly, using Eq. (11). However, it is still necessary to obtain the inverse Laplace transform of the transfer function in order to obtain the Green's function. Using the results of the preceding section, if the transfer function is transformed into its spectral form, the inverse Laplace transform is easier to obtain. The loss of accuracy implied by the truncation of the infinite series in the spectral form is really not that much of a problem in a computer algebraic environment since it is feasible to include an algebraic routine that checks the relative contribution of each additional term in the series and terminates the computation when desired accuracy is obtained. The following section presents the formulation for the algebraic algorithms that were used to transform the transfer function of the combined system into the spectral form by deriving the characteristic parameters and characteristic functions that correspond to the transfer function obtained in Eq. (11).

Algebraic Algorithms for Modified Characteristic Parameters and Functions

Let the transfer function of the baseline system be given by:

$$W_0(x, \xi, p) = \sum_{k=1}^{\infty} \frac{\psi_{0k}(x)\psi_{0k}(\xi)}{p^2 - p_{0k}^2} \frac{1}{\beta_{0k}^2} \tag{12}$$

The purpose of the algebraic algorithms proposed in this paper is to recover from Eq. (11) the transfer function of the combined system in a form similar to Eq. (12). This is done by deriving the appropriate functions and parameters that correspond to the combined system, i.e., the task is to find ψ_{1k}, p_{1k}, and β_{1k}^2, $k = 1,2, \ldots, \infty$ such that the result given in Eq. (11) is equivalent to an expression of the form:

$$W_1(x, \xi, p) = \sum_{k=1}^{\infty} \frac{\psi_{1k}(x)\psi_{1k}(\xi)}{p^2 - p_{1k}^2} \frac{1}{\beta_{1k}^2} \tag{13}$$

Note that, in order to retain the general form of the transfer function a new set of basis function needs to be used. The characteristic parameters of the updated system p_{1k}, $k = 1, 2, \ldots \infty$ are the roots of the equation:

$$\frac{1}{W_c(p)} - W_0(b, b, p) = 0 \tag{14}$$

Computer algebra is used to expand and simplify the left-hand side of Eq. (14) into a ratio of polynomials in p, the numerator of which is then made the argument of an algebraic routine that isolates all its real and complex roots.

Let the kth updated basis function be expanded in terms of the previous set of basis functions:

$$\psi_{1k}(x) = \sum_{n=1}^{\infty} \alpha_{kn}\psi_{0n}(x) \tag{15}$$

The objective of this algorithm is to derive the coefficients α_{kn} using information obtainable from the characteristic parameters and the previous set of basis functions. Direct substitution of Eq. (15) into Eq. (13) gives

$$W_0(x, \xi, p) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha_{kn}\alpha_{km}\psi_{0n}(x)\psi_{0m}(\xi)}{p^2 - p_{1k}^2} \frac{1}{\beta_{1k}^2} \tag{16}$$

Let $Z_n(p)$ be defined as

$$Z_n(p) \equiv \int_{M_2}\int_{M_1} \left[W_0(x, \xi, p) + \frac{W_0(x, b, p)W_0(b, \xi, p)}{1/W_c(p) - W_0(b, b, p)} \right] \times \psi_{0n}(x)\psi_{0n}(\xi)\, \mathrm{d}x\, \mathrm{d}\xi \tag{17}$$

Performing this integration gives

$$Z_n(p) = \frac{1}{\beta_{0n}^2} \left\{ \frac{1}{p^2 - p_{0n}^2} + \frac{W_c(p)(p + p_{1k})}{(p^2 - p_{1k}^2)H_k(p)W_0(b, b, p)} \times \left[\frac{\psi_{0n}(b)}{p^2 - p_{0n}^2} \right]^2 \right\} \tag{18}$$

where

$$H_k(p) = \frac{1}{p - p_{1k}} \left[\frac{1}{W_0(b, b, p)} - W_c(p) \right] \tag{19}$$

Performing the integral operation of Eq. (17) by using Eqs. (11) and (16) and utilizing the orthonormality of the basis vectors ψ_{0n}, we get

$$\int_{M_2}\int_{M_1} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha_{kn}\alpha_{km}\psi_{0n}(x)\psi_{0m}(\xi)}{p^2 - p_{1k}^2} \frac{1}{\beta_{1k}^2} \times \psi_{0n}(x)\psi_{0m}(\xi)\, \mathrm{d}x\, \mathrm{d}\xi = \sum_{k=1}^{\infty} \frac{\alpha_{kn}^2}{p^2 - p_{1k}^2} \frac{1}{\beta_{1k}^2} \tag{20}$$

By virtue of the equivalence of Eqs. (11) and (16), it follows from Eqs. (17) and (20) that

$$\sum_{k=1}^{\infty} \frac{\alpha_{kn}^2}{p^2 - p_{1k}^2} \frac{1}{\beta_{1k}^2} = Z_n(p) \tag{21}$$

One way of obtaining α_{kn} from Eq. (21) is to express $Z_n(p)$ as a ratio of polynomial functions of p:

$$Z_n(p) = \frac{{}_N Z_n(p)}{{}_D Z_n(p)} \tag{22}$$

where ${}_NZ_n(p)$ and ${}_DZ_n(p)$ are the numerator and denominator of the expression $Z_n(p)$, respectively. Let

$$ {}_DZ_{kn}(p) = \frac{{}_DZ_n(p)}{(p^2 - p_{1k}^2)} \tag{23} $$

then,

$$ \frac{\alpha_{kn}^2}{\beta_{1k}^2} = \lim_{p \to p_{1k}} \frac{{}_NZ_n(p)}{{}_DZ_{kn}(p)} \tag{24} $$

If all of the functions of p appearing in Eqs. (18) and (19) are expressed as ratios of polynomials in p, i.e.,

$$ \begin{aligned} W_c(p) &= {}_NW_c(p)/{}_DW_c(p) \\ W_0(b, b, p) &= {}_NW_0(p)/{}_DW_0(p) \\ H_k(p) &= {}_NH_k(p)/{}_DH_k(p) \end{aligned} \tag{25} $$

it follows that

$$ \begin{aligned} {}_NZ_n(p) &= (p^2 - p_{1k}^2)(p^2 - p_{0n}^2){}_DW_c(p){}_NW_0(p){}_NH_k(p) \\ &\quad + (p + p_{1k})\psi_{0n}(b)^2{}_NW_c(p){}_DW_0(p){}_DH_k(p) \end{aligned} \tag{26} $$

$$ {}_DZ_n(p) = (p^2 - p_{1k}^2)(p^2 - p_{0n}^2)^2\ {}_DW_c(p){}_NH_k(p) \tag{27} $$

so that

$$ {}_DZ_{kn}(p) = (p^2 - p_{0n}^2)^2\ {}_DW_c(p){}_NW_0(p){}_NH_k(p) \tag{28} $$

which gives

$$ \frac{\alpha_{kn}^2}{\beta_{1k}^2} = \frac{2p_{1k}\psi_{0n}(b)^2 W_c(p_{1k})}{(p_{1k}^2 - p_{0n}^2)^2 W_0(b, b, p_{1k}) H_k(p_{1k})} \frac{1}{\beta_{0n}^2} \tag{29} $$

If the denominator of Eq. (29) does not vanish, then its computation poses no problems. Equation (14) ensures that $H_k(p_{1k})$ does not vanish. The implication of p_{1k} having the same value as p_{0n} is simply that the kth basis function of the modified system is identical to the nth basis function of the baseline system. As for the vanishing of $W_0(b, b, p_{1k})$, which can occur only if the attachment point of the substructure happens to be a fixed point on the baseline structure, it is obvious that the attached substructure will not affect the dynamics of the system.

The basis functions $\psi_{1k}(x)$ of the modified structure are usually normalized such that

$$ \int_0^L \psi_{1k}(x)^2\, dx = 1 \tag{30} $$

which implies that

$$ \sum_{n=1}^{\infty} \alpha_{kn}^2 = 1 \tag{31} $$

This condition leads to the determination of β_{1k}^2 as

$$ \beta_{1k}^2 = \frac{1}{\sum_{n=1}^{\infty}(\alpha_{kn}^2/\beta_{1k}^2)} \tag{32} $$

In general, the coefficients α_{kn} are complex valued. The result obtained from Eq. (29) subject to Eq. (31) only yields the moduli of α_{kn}. In order to determine the phase of α_{kn} (i.e., the real and imaginary parts of α_{kn}), it is necessary to invoke additional requirements on α_{kn}. For systems without dissipation, it is reasonable to require the modified functions to describe the unforced motions of the system, i.e.,

$$ \psi_{1k}(x) = W_0(x, b, p_{1k}) W_c(p_{1k}) \psi_{1k}(b) \tag{33} $$

Upon expanding Eq. (33) in terms of the baseline functions and the transformation coefficients, we get

$$ \alpha_{kn} = \sum_{m=1}^{\infty} \frac{W_c(p_{1k})}{\beta_{0n}^2(p_{1k}^2 - p_{0n}^2)} \alpha_{km}\psi_{0m}(b)\psi_{0n}^*(b) \tag{34} $$

Define an objective function:

$$ h_k \equiv \sum_{n=1}^{\infty} \left\| \alpha_{kn} - \frac{W_c(p_{1k})\psi_{0n}^*(b)}{\beta_{0n}^2(p_{1k}^2 - p_{0n}^2)} \sum_{m=1}^{\infty} \alpha_{km}\psi_{0m}(b) \right\|^2 \tag{35} $$

Any convenient routine can be used to seek the phases of the α such that h_k is a minimum. For undamped systems, an absolute minimum of zero (within limits of machine error) should be achievable, whereas for damped systems, the minimum of h_k may not be zero. However, this method suggests that the answer should approach the undamped case when the modified characteristic functions are real.

Example

A cantilever beam with a spring and mass attached to the free end is analyzed for different values of the spring constants and masses (Fig. 1). This example is selected because the results can be compared to results obtained by Young.[5] The location of the spring-mass attachment was chosen to be $b/L = 1.0$. The spring stiffness K was chosen to have the ratios $K/Kb = 0.101$, 1.010, 10.101, and 50.505; where $K_b = 3EI/L^3$ is the stiffness of the cantilever beam. The mass M was selected to have the ratio $M/(\rho AL) = 0$, 0.1, and 1; where ρ is the density of the material, A the cross-sectional area of the beam, and L its length.

Five combinations of spring-mass attachments as shown in Table 1 were used. The results were compared to results of a finite element analysis using three different meshes as well as results obtained by Young.[5]

The comparison of natural frequencies generated by the proposed method, the finite element method, and the method outlined by Young[5] are shown in Tables 2–6. The nondimensional values shown in these tables are the ratio between the frequencies calculated and the fundamental natural frequency of the cantilever beam without attachments.

The expression in Eqs. (36–38) are the algebraic results derived for the first, second, and third modes of the modified structure 5, based on the spectral expansion of Eq. (15):

$$ \begin{aligned} \psi_{11}(x) &= 2.027e^{-4}\,[-\sin(0.14x) + \cos(0.14x) - e^{-0.14x}] \\ &\quad - 5.645e^{-4}\,[-\sin(0.11x) + \cos(0.11x) - e^{-0.11x}] \\ &\quad + 3.267e^{-6}\,\{-1288\,[-\sinh(0.079x) + \sin(0.079x)] \\ &\quad + 1289\,[-\cosh(0.079x) + \cos(0.079x)]\} \\ &\quad - 1.004e^{-3}\,\{-54.636\,[-\sinh(0.046x) + \sin(0.047x)] \end{aligned} $$

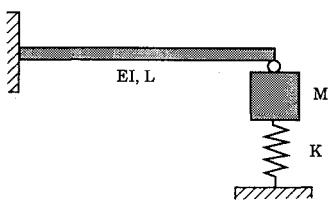

Fig. 1 Schematic of beam with spring-mass attachment.

Table 1 Spring constants and masses of modified structures

Modified structure	K/K	$M/(\rho AL)$
1	0.101	0.1
2	1.010	0.1
3	0.101	1
4	10.101	0
5	50.505	0

Table 2 Comparison of frequencies for modified structure 1 recovered from the transfer function, Young's method, and the finite element method with different mesh definition

Mode	Finite element, 2 elements	Finite element, 10 elements	Finite element, 50 elements	Young's method	Proposed method
1	0.820	0.886	0.888	0.886	0.885
2	4.528	5.479	5.532	N/A	5.516
3	240	15.63	16.40	N/A	15.87

Table 3 Comparison of frequencies for modified structure 2 recovered from the transfer function, Young's method, and the finite element method with different mesh definition

Mode	Finite element, 2 elements	Finite element, 10 elements	Finite element, 50 elements	Young's method	Proposed method
1	1.104	1.187	1.190	1.190	1.188
2	4.69	5.51	5.56	N/A	5.55
3	240	15.64	16.43	N/A	15.88

Table 4 Comparison of frequencies for modified structure 3 recovered from the transfer function, Young's method, and the finite element method with different mesh definition

Mode	Finite element, 2 elements	Finite element, 10 elements	Finite element, 50 elements	Young's method	Proposed method
1	0.456	0.467	0.467	0.470	0.465
2	4.315	4.635	4.643	N/A	4.644
3	240	246	11.6	N/A	14.7

Table 5 Comparison of frequencies for modified structure 4 recovered from the transfer function, Young's method, and the finite element method with different mesh definition

Mode	Finite element, 2 elements	Finite element, 10 elements	Finite element, 50 elements	Young's method	Proposed method
1	2.825	2.867	2.869	2.881	2.873
2	4.921	7.021	7.142	N/A	7.134
3	240	17.38	17.89	N/A	17.85

Table 6 Comparison of frequencies for modified structure 5 recovered from the transfer function, Young's method, and the finite element method with different mesh definition

Mode	Finite element, 2 elements	Finite element, 10 elements	Finite element, 50 elements	Young's method	Proposed method
1	4.029	3.985	3.982	3.982	3.990
2	7.430	9.923	9.993	N/A	10.117
3	240	18.65	19.24	N/A	19.37

Table 7 Basis functions used to derive the mode shapes of the modified structure

n_1	ω_{0n_1}	A	B	C	D	$\psi_{0n_1}(x)$
1	3.52	0.024	4.138	0.019	3.038	$A\{B[\cos(Cx)$
2	22.03	0.002	53.65	0.047	54.64	$-\cosh(Cx)]$
3	61.7	$7.67e^{-5}$	1290	0.079	1289	$D[\sin(Cx)$
						$-\sinh(Cx)]\}$
4	121	0.102	—	0.110	—	$A[\cos(Cx)$
5	200	0.102	—	0.141	—	$-\sin(Cx)$
6	299	0.105	—	0.173	—	$-e^{-Cx}]$

$$+53.645\,[-\cosh(0.047x)+\cos(0.047x)]\}$$
$$-0.02033\,\{-3.04\,[-\sinh(0.019x)+\sin(0.012x)]$$
$$+4.14\,[-\cosh(0.019x)+\cos(0.012x)]\} \quad (36)$$

$$\psi_{12}(x) = -3.782e^{-4}\,[-\sin(0.17x)+\cos(0.17x)-e^{-0.17x}]$$
$$+8.650e^{-4}\,[-\sin(0.14x)+\cos(0.14x)-e^{-0.14x}]$$
$$-2.533e^{-3}[-\sin(0.11x)+\cos(0.11x)-e^{-0.11x}]$$
$$+1.932e^{-5}\,\{-1288\,[-\sinh(0.079x)+\sin(0.079x)]$$
$$+1289\,[-\cosh(0.079x)+\cos(0.079x)]\}$$
$$+1.530e^{-3}\,\{-54.636\,[-\sinh(0.046x)+\sin(0.047x)]$$
$$+53.645\,[-\cosh(0.047x)+\cos(0.047x)]\}$$
$$-0.01238\,\{-3.04\,[-\sinh(0.019x)+\sin(0.012x)]$$
$$+4.14\,[-\cosh(0.019x)+\cos(0.012x)]\} \quad (37)$$

$$\psi_{13}(x) = 4.749e^{-4}\,[-\sin(0.17x)+\cos(0.17x)-e^{-0.17x}]$$
$$-1.144e^{-3}\,[-\sin(0.14x)+\cos(0.14x)-e^{-0.14x}]$$
$$+4.079e^{-3}\,[-\sin(0.11x)+\cos(0.11x)-e^{-0.11x}]$$
$$+7.411e^{-5}\,\{-1288\,[-\sinh(0.079x)+\sin(0.079x)]$$

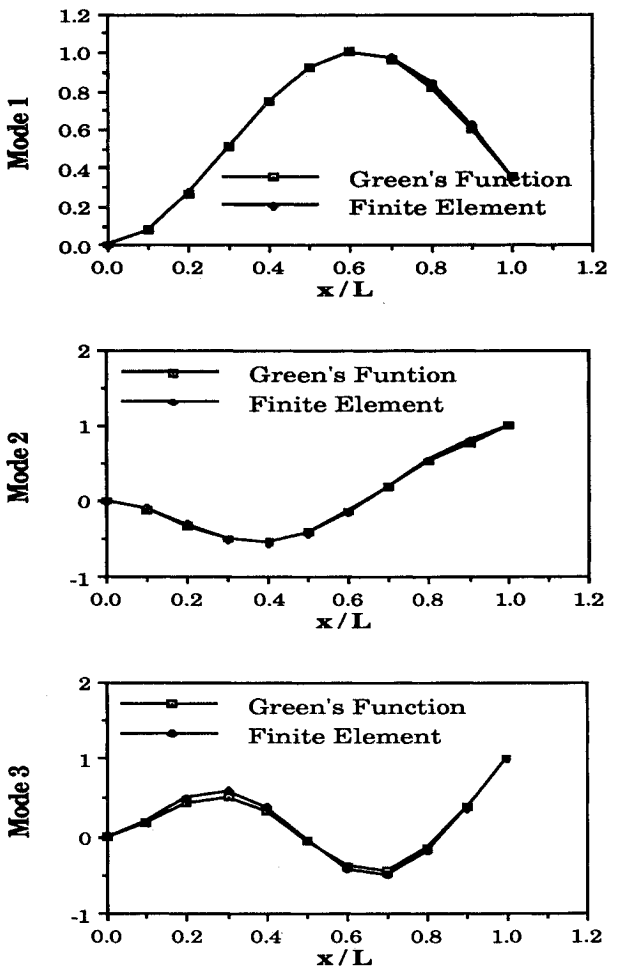

Fig. 2 Comparison of mode shapes predicted by the proposed and the finite element methods.

$$+1289\,[-\cosh(0.079x)+\cos(0.079x)]\}$$

$$-3.529e^{-4}\{-54.636\,[-\sinh(0.046x)+\sin(0.047x)]$$

$$+53.645\,[-\cosh(0.047x)+\cos(0.047x)]\}$$

$$+0.004103\{-3.04\,[-\sinh(0.019x)+\sin(0.012x)]$$

$$+4.14\,[-\cosh(0.019x)+\cos(0.012x)]\} \qquad (38)$$

These expressions are plotted and compared to the result of the finite element analysis using 50 elements in Fig. 2. It can be observed that the proposed method predicts the mode shapes and frequencies of the modified structure accurately. It can therefore be concluded that Eq. (13) is the correct transfer function of the modified structure. For the present example, the mode shapes of the modified structure are assumed to be spanned by the natural modes of the baseline structure. A list of the mode shapes used are shown in Table 7, which was obtained for a uniform cantilevered Euler-Bernoulli beam with $EI = 10^8$, $\rho A = 1$, and $L = 100$ in consistent units.

Conclusions

This paper has presented a method that utilizes computer algebra to derive the Green's function of a combined dynamical system consisting of a distributed parameter baseline structure to which a discrete substructure has been attached. The general form of the Green's functions of the systems governed by Sturm-Liouville type differential equations has been used to recover the characteristic functions and characteristic parameters of the combined system from the expressions obtained for the Green's functions using methods based on a general theory of interconnected distributed parameter systems. Examples of a uniform cantilever beam with discrete spring-mass attachments were used to confirm the agreement of the extracted parameters and functions with other methods such as the finite element method.

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